JOURNAL OF ALGEBRA 164, 642–655 (1994)

Brauer's $k(B)$-Conjecture for Solvable 3′-Groups

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Communicated by George Glauberman

Received March 18, 1992

1. INTRODUCTION

Brauer's famous $k(B)$-conjecture asserts that the number of irreducible complex characters in a p-block B of a finite group G is at most the order of the defect group of B. Using arguments due to Fong, Nagao (see [10]) showed that, when G is p-solvable, the problem may be reduced to the so-called $k(GV)$-conjecture. The latter asserts that if V is an elementary abelian p-group on which the p'-group G acts faithfully and irreducibly, then the number of conjugacy classes of the semidirect product GV is at most $|V|$. The main progresses to treat this problem were made by Knörr in [8], where he developed highly original techniques. They led to the solution when G is of odd order, which was obained independently by Gluck and Knörr (see [3]). More recently, Knörr proved the following important result (see Corollary of [9]):

Let G be a p'-group acting faithfully on an elementary abelian p-group V. Assume that there exists a vector $v \in V$ such that $V_{C_G(v)}$ is a permutation module. Then the number of conjugacy classes of GV is at most $|V|$.

The purpose of this paper is to use this result to show that Brauer's conjecture holds for solvable 3′-groups. By the previous comments, this is a corollary of the following theorem:

MAIN THEOREM. *Let G be a solvable 3′-group and let V be a faithful KG-module, where* $\mathrm{char}(K) \neq 3$ *and* $\mathrm{char}(K) \nmid |G|$. *Then there exists $v \in V$ such that $V_{C_G(v)}$ is a permutation module.*

It is easy to see that, when we treat to prove the main theorem, we may assume that V is primitive. Thus we must prove that a faithful and primitive KG-module V contains a vector v such that $V_{C_G(v)}$ is a permutation module. This happens very often when G is solvable but not always (see Example 4.1). To find all possible counterexamples is a very long and

0021-8693/94 $6.00

complicated task, however. Thus we limit our study to the 3'-case although several reductions and techniques may be interesting in the general solvable case.

The notations of this paper follows [7] and the ATLAS [1] for the orthogonal and symplectic groups. We denote by $V \otimes FG$ the extension of the KG-module V, where F is a field extension of K. $A * B$ denotes the central product of A and B.

This research has been partially supported by the DGICYT by Grant PB 90-923.

2. The Quasiprimitive Case

As we have commented in the Introduction, the problem is naturally reduced to the case in which V is primitive. But we are able to prove a slightly stronger assertion. We may reach the hypotheses of the next result:

THEOREM 2.1. *Let G be a solvable 3'-group satisfying the following:*

(i) *$F(G) = Z(G) * E$, where $Z(G) = Z(F(G))$ is cyclic, E is normal in G, and each Sylow subgroup of E is extraspecial. (It is possible that $E = 1$.)*

(ii) *E does not contain any extraspecial subgroup of order 8 normal in G.*

(iii) *The $G/F(G)$-module $E/Z(E)$ is faithful and completely reducible.*

Let V be a KG-module, $3 \neq \mathrm{char}(K) \nmid |G|$, such that $[V, z] = V$ if $1 \neq z \in Z(G)$. Let F be the algebraic closure of K. Then there exists $v \in V$ such that $W_{C_G(v)}$ is a permutation module for each irreducible constituent W of $V \otimes FG$.

We need several results contained in [6]. We summarize them in the following lemma.

LEMMA 2.2. *Let G be a solvable group having a normal p-subgroup P such that $Z(P) \leqslant Z(G)$. Assume that P is the central product of a cyclic group and an extraspecial group. Let ϑ be a faithful irreducible character of P such that ϑ is extendible to G and let χ be an extension of ϑ to G.*

An element $x \in G$ is said to be good when $C_{P/Z(P)}(x) = C_P(x)/Z(P)$. Then:

(i) *$\vartheta(x) = 0 = \chi(x)$ if $x \in P - Z(P)$.*

(ii) *$\chi(x) = 0$ if x is not good.*

(iii) *$|\chi(x)|^2 = |C_{P/Z(P)}(x)|$ when x is good and $\langle x \rangle \cap P \leqslant Z(P)$.*

Proof. (i) is well known. Observe that our definition of good elements agrees with Definition 3.1 of [6]. Now Lemma 2.4 (b) of [6] ensures that

$\chi(x)=0$ when x is not good. If x is good and $\langle x \rangle \cap P \leqslant Z(P)$, then put $U=\langle x \rangle Z(P)$. Applying Theorem 3.5 of [6] to $\langle x \rangle P$ and $\chi_{\langle x \rangle P}$, we get our claim.

Our first result studies a particular configuration which appears several times in our arguments.

LEMMA 2.3. *Let G be a solvable group such that $Z(G)$ is cyclic and $F(G) \simeq Z(G) * Q_8 * D_8$. Suppose that $G/F(G)$ contains a subgroup of order 5. Let W be a KG-module,* $\operatorname{char}(K) \nmid |G|$, *such that $[W, z]=W$ if $1 \neq z \in Z(G)$. Suppose that x is an element of order 2 or 4 of G such that $\langle x \rangle \cap F(G)=1$. Then $W_{\langle x \rangle}$ is a sum of copies of the regular module.*

Proof. The Noether–Deuring theorem (see Theorem VII.1.22 of [5]) ensures that we may assume that K is algebraically closed. As $\operatorname{char}(K)$ does not divide $|G|$, we may suppose that $K=\mathbb{C}$. Clearly we may assume that W is irreducible. Let χ be the character of G afforded by W. We show that $\chi(x)=0$ when x is as in the statement. This yields our claim.

First we show that it is enough to consider the case $F(G) \simeq Q_8 * D_8$. Put $\bar{G}=G/F(G)$. Observe that $\bar{G}$ is isomorphic to a subgroup of $GO_4^-(2)$. Now, as $GO_4^-(2)$ is isomorphic to the group S_5, we have that $\bar{G}$ is a subgroup of $C_4 \ltimes C_5$, where the action is faithful. Let $\bar{R}$ be a Sylow 5-subgroup of $\bar{G}$. Put $F_0/F(G)'=[F(G)/F(G)', \bar{R}]$. Now $F_0 \simeq Q_8 * D_8$ and F_0 is normal in G. Let X be the preimage in G of $[G/F_0, x]$. Then X/F_0 is cyclic of order 5 and X is normal in G. We may suppose that $G=\langle x \rangle X$ and therefore $F(G) \simeq Q_8 * D_8$.

Let ϑ be the faithful and irreducible character of degree 4 of $F(G)$. By Corollaries 6.28 and 11.22 of [7], we have an extension ψ of ϑ to G. Furthermore $\chi=\psi\mu$, $\mu \in \operatorname{Irr}(\bar{G})$. If $\psi(x)=0$, then $\chi(x)=0$. Hence may assume that $\chi=\psi$. Then $\chi(1)=4$.

Suppose first that x is an involution. Take a Sylow 5-subgroup R of G. Consider $H=\langle x \rangle(RF(G))$. Now a Sylow 2-subgroup of $N_H(R)$ is either cyclic of order 4 or elementary abelian. Take $t \in N_H(R)-Z(G)$. Then $t^2 \in Z(G)$. By Lemma 2.2, we must show that x is not good. We prove first that t is not good. By applying to the group $RF(G)$ Theorem V.17.13 of [5], we have that χ_R is the sum of four different irreducible characters of R. As t permutes the irreducible constituents of χ_R, we get that χ_R is the sum of the nontrivial characters of R. Now, if t is an involution, then $\chi_{\langle t \rangle}$ is the sum of two copies of the regular module. If t is of order 4, then $\chi_{\langle t \rangle}=2\lambda+2\bar{\lambda}$, where $\lambda(t)=i$. In both cases, $\chi(t)=0$ and t is not good.

Observe that there are exactly 10 cosets of $Z(G)$ in $F(G)$ consisting of elements of order 4 and exactly five nontrivial cosets consisting of involutions. The action of $\langle t \rangle R$ on these cosets yields that t fixes exactly two cosets consisting of elements of order 4 and exactly one nontrivial coset consisting of involutions. Then the preimage C of $C_{F(G)/Z(G)}(t)$ in G is isomorphic to $C_4 \times C_2$. But $x = ta$, where $a \in F(G)$ and t normalizes a Sylow 5-subgroup of G. As x is an involution and $t^2 \in Z(G)$, we get that the coset $aZ(G)$ is fixed by t. Thus $a \in C$. As t is not good, then t does not centralize C. As C is abelian, we get that C is not centralized by x. As $C/Z(G) = C_{F(G)/Z(G)}(x)$, we conclude that x is not good, as claimed. Thus the lemma is proved when x is an involution.

Assume that $o(x) = 4$ and $\langle x \rangle \cap F(G) = 1$. If x is not good, then x^3 is not good. As x^2 is not good by the previous paragraph, we have that χ vanishes on all nontrivial elements of $\langle x \rangle$ and then $\chi_{\langle x \rangle}$ is a sum of copies of the regular module, as claimed. We suppose that x is good and we derive a contradiction. We have $\chi(x^2) = 0$. By Lemma 2.2, we have $|\chi(x)|^2 = |\chi(x^3)|^2 = 2$. Let λ be an irreducible character of $\langle x \rangle$. Now

$$a = \langle \chi_{\langle x \rangle}, \lambda \rangle = \tfrac{1}{4}\,(4 + \chi(x)\,\bar{\lambda}(x) + \chi(x^3)\,\bar{\lambda}(x^3)) = \tfrac{1}{4}\,(4 + b).$$

As $|b| \leqslant 2\sqrt{2}$ and a is an integer, we get $b = 0$. Now $a = 1$ for each λ and $\chi_{\langle x \rangle}$ is the regular module, a contradiction which finishes the proof of the lemma.

Our next result is the analogue of Lemma 1.3 of [2] for 3′-groups. As the proof is very similar, we sketch it only.

LEMMA 2.4. *Let G be a solvable 3′-group such that $Z(G) = Z(F(G))$, $F(G) = Z(G) * E$, where $Z(G)$ is cyclic and the Sylow subgroups of E are extraspecial. Let V be a KG-module,* $\mathrm{char}(K) \nmid |G|$, *such that $[V, z] = V$ if $1 \neq z \in Z(G)$. Then, if $1 \neq x \in G$, we have*

$$\dim_K(C_V(x)) \leqslant 5 \dim_K(V)/8.$$

Proof. We may assume that V is irreducible, $o(x)$ is a prime number p, and $K = \mathbb{C}$. If x lies in $F(G)$, then we get that $\dim_K(C_V(x)) \leqslant \dim_K(V)/2$ as in Lemma 1.3 of [2]. If $x \notin F(G)$ and $[x, O_{p'}(F(G))] \neq 1$, we get $\dim_K(C_V(x)) \leqslant 3 \dim_K(V)/5$ using Hall–Higman Theorem B as in [2]. For the remaining case, put $P = O_p(G)$, $\bar{P} = P/Z(P)$, and $|\bar{P}| = p^{2n}$. We may get a special $\langle x \rangle$-invariant subgroup $\bar{Q}$ of $O_{p,p'}(G)/O_p(G)$ such that $\bar{Q}[x, \bar{Q}]$ and $[x, \Phi(\bar{Q})] = 1$. If $p = 2$, then we may assume that $\bar{Q}$ is of prime order. Applying Theorem IX.2.6 of [5] to the action of $\langle x \rangle \bar{Q}$ on $\bar{P}$, we get $\dim_{GF(p)}(C_{\bar{P}}(x)) \leqslant \dim_{GF(p)}(\bar{P}) - 3$ except when $|\bar{P}| = 2^4$. Assume that $|\bar{P}| \neq 2^4$. Let W be an irreducible $\langle x \rangle$ P-submodule of V. We prove that $\dim_K(C_W(x)) \leqslant 5 \dim_K(W)/8$. This yields our claim. Let χ be the character afforded by W. Now, if $p \neq 2$, then

$$\dim_K(C_W(x)) = \langle \chi_{\langle x \rangle}, 1_{\langle x \rangle} \rangle \leqslant (p^n + p \cdot p^{2n - 3/2})/p \leqslant 2 \dim_K(W)/5.$$

If $p=2$ then, as x is an involution, we have that $\chi(x)$ is a rational integer. Remember that $|\bar{P}| \neq 2^4$. Now

$$\dim_K(C_W(x)) \leqslant (2^n + 2^{n-2})/2 = 5 \dim_K(W)/8.$$

Assume therefore that $|\bar{P}| = 2^4$. Then $\bar{Q}$ is of order 5 and P is the central product of $Z(P)$ and $Q_8 * D_8$. Let Q be the preimage of $\bar{Q}$ in G. Applying Lemma 2.3 to the action of $\langle x \rangle Q$ on V, we get $\dim_K(C_V(x)) = \dim_K(V)/2$. The lemma is proved.

The last result we need is due to Wolf (see Theorem 3.1 of [11]):

LEMMA 2.5. *Let G be a solvable group and let $0 \neq V$ be a faithful and completely reducible KG-module. Then $|G| \leqslant |V|^{9/4}/(24)^{1/3}$.*

Proof of Theorem 2.1. We may assume that V is an irreducible KG-module. If we prove that, for example, V contains a regular G-orbit, then the result will be clear in this case. Put $\bar{G} = G/F(G)$ and $\bar{E} = E/Z(E)$. We make the following

Remarks. (1) Suppose that E is a p-group, p a prime. As $\bar{E}$ is a faithful and completely reducible $\bar{G}$-module, we have that $F(\bar{G})$ is a p'-group.

(2) Suppose that E is a p-group. If $\bar{A}$ is a normal p'-subgroup of $\bar{G}$, then the preimages, in E of $[\bar{E}, \bar{A}]$ and $C_{\bar{E}}(\bar{A})$ are extraspecial. Now (ii) ensures that $\dim C_{\bar{E}}(\bar{A}) \neq 2 \neq \dim([\bar{E}, \bar{A}])$.

Put $e^2 = |E/Z(E)|$. We first show that when e is large enough, then V contains a regular G-orbit. We have

$$|G| = |\bar{G}|\, e^2\, |Z(G)|.$$

Now (iii) and Lemma 2.5 yield that, when $e > 1$, then

$$|\bar{G}| \leqslant e^{9/2}/(24)^{1/3}.$$

Let U be an irreducible $Z(G)$-submodule of V. Clearly $|Z(G)| \mid |U| - 1$. Furthermore $|V| \geqslant |U|^e$. Now $|\bigcup_{1 \neq x \in G} C_V(x)| < |G|\, |V|^{5/8}$ by Lemma 2.4. If we show that

$$|G| \leqslant |V|^{3/8}, \tag{$*$}$$

then clearly $V \neq \bigcup_{1 \neq x \in G} C_V(x)$ and V contains a regular G-orbit. Then it is enough to show that

$$e^{13/2} \leqslant (24)^{1/3}\, |U|^{(3e/8)-1}. \tag{$**$}$$

As U is a faithfull $Z(G)$-module and $\mathrm{char}(K) \neq 3$, we get $|U| \geqslant 5$. Then it is enough to show that

$$e^{13/2} \leqslant (24)^{1/3}\, 5^{(3e/8)-1}.$$

If $e \geqslant 64$, then the latter inequality holds. We consider the remaining cases. Assume first that e is a power of 2. If $e = 1$, then G is cyclic and clearly V contains a regular G-orbit. By (ii), we have $e \neq 2$.

Case $e = 4$. As $D_8 * D_8$ does not admit any $\{2, 3\}'$-automorphism, we get that $F(G) \simeq Z(G) * Q_8 * D_8$. Now $\bar{G}$ is isomorphic to a subgroup of $GO_4^-(2)$, which is isomorphic to the symmetric group S_5. As 5 divides $|F(\bar{G})|$ and a Sylow 5-subgroup of S_5 is self-centralizing, we get that $\bar{G}$ is isomorphic to a subgroup of $C_4 \ltimes C_5$, where the action is faithful. Let Q be a Sylow 5-subgroup of G. Next we show that $QF(G)$ has a regular orbit on V. Applying to the group $QF(G)$ Theorem V.17.13 of [5], we deduce that $\dim_K(C_V(x))$ is at most $\dim_K(V)/4$ when x is a noncentral element of order 5 of G. We have at most $16 \cdot 5$ noncentral subgroups of order 5 in G (since, possibly, $5 \mid |Z(G)|$). Now $F(G)$ contains at most $2 \cdot 15$ noncentral involutions. Thus

$$\left| \bigcup_{1 \neq x \in QF(G)} C_V(x) \right| \leqslant 80\, |V|^{1/4} + 30\, |V|^{1/2} < |V| \qquad \text{since} \quad |V| \geqslant 7^4.$$

Now let $v \in V$ be a generator of a regular $QF(G)$-orbit. Then $C_G(v)$ is isomorphic to a subgroup of C_4. Applying Lemma 2.3, we get that $W_{C_G(v)}$ is a sum of copies of the regular module for each irreducible constituent W of $V \otimes FG$.

Case $e = 8$. We have two possibilities:

(a) $F(G) \simeq Z(G) * D_8 * D_8 * D_8$. Here $\bar{G}$ is isomorphic to a subgroup of $GO_6^+(2)$, which is isomorphic to the symmetric group S_8. By Remark (2) we have that 7 divides $|F(\bar{G})|$. As a 7-Sylow of S_8 is self-centralizing, we have $|F(\bar{G})| = 7$ and $\bar{G}$ is isomorphic to a subgroup of $C_2 \ltimes C_7$. Let Q be a Sylow 7-subgroup of G. We show that V contains a regular $QF(G)$-orbit.

Observe that G has at most $64 \cdot 7$ noncentral subgroups of order 7. Using Theorem V.17.13 of [5] we get that $\dim_K(C_V(x))$ is at most $\dim_K(V)/4$ when x is a noncentral element of order 7 of G. Furthermore, $F(G)$ contains at most $2 \cdot 63$ noncentral involutions. Then

$$\left| \bigcup_{1 \neq x \in QF(G)} C_V(x) \right| \leqslant 64 \cdot 7 \cdot |V|^{1/4} + 2 \cdot 63 \cdot |V|^{1/2} < |V|, \qquad \text{since} \quad |V| \geqslant 5^8.$$

Let $v \in V$ be a generator of a regular $QF(G)$-orbit. Put $C_G(v) = \langle x \rangle$. If x is nontrivial, then x is an involution in $G - F(G)$. The action of $\bar{G}$ on $\bar{E}$

shows that $|C_E(x)| = 2^3$. Let W be an irreducible consistuent of $V \otimes FG$. We may suppose that F is the complex field. Let χ be the character of G afforded by W and let ψ be an irreducible constituent of $\chi_{\langle x \rangle X}$, where X is a Sylow 2-subgroup of $F(G)$. If x is a good element, then Lemma 2.2 yields that $|\psi(x)|^2 = 2^3$, a contradiction, since $\psi(x)$ is a rational integer. Hence x is not good and Lemma 2.2 yields $\psi(x) = 0$. As this is true for each irreducible constituent of $\chi_{\langle x \rangle X}$, we get that $\chi(x) = 0$ and $W_{\langle x \rangle}$ is a sum of copies of the regular module.

(b) $F(G) \simeq Z(G) * Q_8 * D_8 * D_8$. Here $\bar{G}$ is a subgroup of $GO_6^-(2)$, whose order is $2^7 \cdot 3^4 \cdot 5$. Then 5 divides $|F(\bar{G})|$ and we contradict Remark (2).

Case $e = 16$. (a) $F(G) \simeq Z(G) * D_8 * D_8 * D_8 * D_8$. Here $\bar{G}$ is isomorphic to a subgroup of $GO_8^+(2)$, whose order is $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$. If 7 divides $|F(\bar{G})|$ we contradict Remark (2). We deduce that $5 \mid |F(\bar{G})|$. If $F(\bar{G})$ is of order 5, then $|\bar{G}| \leqslant 20$ and

$$|V|^{3/8}/|G| \geqslant |U|^6/|G| \geqslant 7^5/20 \cdot 2^8 > 1.$$

Then $(*)$ holds and V contains a regular G-orbit.

If $F(\bar{G})$ is of order 25 then, as a Sylow 5-subgroup of $\bar{G}$ is elementary abelian and $\bar{G}$ is a $3'$-group, then $\bar{G}$ is isomorphic to a subgroup of $(C_4 \ltimes C_5) \wr C_2$. The intersection of $\bar{G}$ with the base group of this wreath product is a subgroup $\bar{G}_0$ of index at most 2 in $\bar{G}$. Let G_0 be the preimage of $\bar{G}_0$ in G. We have $\bar{E} = \bar{P}_1 \oplus \bar{P}_2$, where $|\bar{P}_1| = |\bar{P}_2| = 2^4$ and $\bar{G}_0$ acts componentwise on $\bar{P}_1 \oplus \bar{P}_2$.

If $x \in G_0 - F(G)$ is an involution and $\bar{x}$ is the image of x in $\bar{G}$, then we may find a subgroup $\bar{Q}$ of order 5 of $\bar{G}$ such that $\bar{x}$ normalizes $\bar{Q}$, $\bar{x}$ does not centralize $\bar{Q}$ and $\bar{Q}$ centralizes either $\bar{P}_1$ or $\bar{P}_2$. Let X be the preimage of the group $\bar{Q}$ in G.

Now, if Q is a Sylow 5-subgroup of X, then $[Q, E]$ is extraspecial of order 2^5 and $\langle x \rangle$-invariant. Also $Q[Q, E]$ is normalized by $\langle x \rangle$. We apply Lemma 2.3 to $H = \langle x \rangle (Q[Q, E])$. We conclude that $\dim_K(C_V(x))$ is at most $\dim_K(V)/2$. Theorem V.17.13 of [5] ensures that the same inequality holds when x is of order 5. Thus it holds for every nontrivial element of G_0.

If $\dim_K(V) > e$, then either $\dim_K(U) > 1$ or $|V| \geqslant |U|^{2e}$. In the first case, we have

$$|V|^{3/8}/|G| \geqslant |U|^6/|G| \geqslant |U|^5/2^5 \cdot 5^2 \cdot 2^8 \geqslant 7^{10}/2^5 \cdot 5^2 \cdot 2^8 > 1.$$

In the second, we obtain

$$|V|^{3/8}/|G| \geqslant |U|^{12}/|G| \geqslant |U|^{11}/2^5 \cdot 5^2 \cdot 2^8 \geqslant 7^{11}/2^5 \cdot 5^2 \cdot 2^8 > 1.$$

Then $(*)$ holds and V contains a regular orbit in both cases. Assume hence that $\dim_K(V) = e$. Then $V \otimes FG$ is an irreducible FG-module. Define $S = \{x \in G \mid \dim_K(C_V(x)) \leqslant \dim_K(V)/2\}$. We know that $G_0 - \{1\} \subseteq S$. We have

$$\left| \bigcup_{x \in S} C_V(x) \right| \leqslant 2^5 \cdot 5^2 \cdot 2^8 \cdot |Z(G)| \cdot |V|^{1/2}.$$

But $|V|^{1/2} \geqslant |U|^8 \geqslant |Z(G)| \cdot 7^7$. As $2^{13} \cdot 5^2 < 7^7$, we may find a vector v in V which is not centralized by any element of S. Now, if $C_G(v)$ is nontrivial, then $C_G(v) = \langle x \rangle$, where x is an involution not in S. Hence $\dim_K(C_V(x)) > \dim_K(V)/2$. As $V \otimes FG$ is irreducible, then $W = V \otimes FG$ in this case. Now $W_{\langle x \rangle}$ is a sum of copies of the regular module and the trivial module. Hence $W_{\langle x \rangle}$ is a permutation module in this case.

(b) $F(G) \simeq Z(G) * Q_8 * D_8 * D_8 * D_8$. Here $\bar{G}$ is isomorphic to a subgroup of $GO_8^-(2)$, whose order is $2^{13} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$. As in case (a), we have that $7 \nmid |F(\bar{G})|$. Suppose that $17 \mid |F(\bar{G})|$. The ATLAS, [1] shows that a Sylow 17-subgroup of $GO_8^-(2)$ is self-centralizing. Then $|F(\bar{G})| = 17$ and $\bar{G}$ is isomorphic to a subgroup of $C_{16} \ltimes C_{17}$. We show that V contains a regular G-orbit.

If $x \in G - F(G)$ is an involution and a is an element of $F(G)$ such that xa is also an involution, then $o(a)$ divides 4 and $aF(G)'$ lies in $C_{F(G)/F(G)'}(x)$. The action of $\bar{G}$ on $\bar{E}$ yields that $|C_{\bar{E}}(x)| = 2^4$. Hence $G - F(G)$ contains at most $17 \cdot 2^4 \cdot 2 \cdot 2$ involutions. Furthermore $F(G)$ contains at most $255 \cdot 2$ noncentral involutions. The number of noncentral subgroups of order 17 of G does not exceed $2^8 \cdot 17$. Thus

$$\left| \bigcup_{1 \neq x \in G} C_V(x) \right| \leqslant (2^6 \cdot 17 + 255 \cdot 2 + 2^8 \cdot 17)\, |V|^{5/8} < |V| \qquad \text{since} \quad |V| \geqslant 5^{16}$$

Hence V contains a regular G-orbit.

Suppose now that 17 does not divide $|F(\bar{G})|$. Now $F(\bar{G})$ is of order 5 and $\bar{G}$ has order at most 20. But then

$$|V|^{3/8}/|G| \geqslant |U|^6/20 \cdot 2^8 \cdot |Z(G)| \geqslant 7^5/20 \cdot 2^8 > 1.$$

Then $(*)$ holds and V contains a regular G-orbit in this case.

Case $e = 32$. (a) $F(G) \simeq Z(G) * D_8 * D_8 * D_8 * D_8 * D_8$. Here $\bar{G}$ is isomorphic to a subgroup of $GO_{10}^+(2)$, whose order is $2^{31} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$. If 17 divides $|F(\bar{G})|$, we contradict remark (2). If 31 divides $|F(\bar{G})|$, then the ATLAS [1] reveals that a Sylow 31-subgroup of $\bar{G}$ is self-centralizing and then $|F(\bar{G})| = 31$. Hence $|\bar{G}| \leqslant 30 \cdot 31$. Now

$$|V|^{3/8}/|G| \geqslant |U|^{12}/|G| \geqslant 5^{11}/30 \cdot 31 \cdot 2^{10} > 1.$$

Then $(*)$ holds and V contains a regular G-orbit. Suppose that $|F(\bar{G})|$ is not divisible by 31. Then $F(\bar{G})$ is a $\{5, 7\}$-group. As $F(\bar{G})$ is self-centralizing, we get that $F(\bar{G})$ contains a Hall $2'$-subgroup of $\bar{G}$. Now hypotheses (ii) and (iii) yield that $C_E(F(\bar{G})) = 1$. This implies that 35 divides $|\bar{G}|$. But the ATLAS [1] yields that $GO_{10}^{+}(2)$ does not contain any element of order 35, a contradiction.

(b) $F(G) \simeq Z(G) * Q_8 * D_8 * D_8 * D_8 * D_8$. Here $\bar{G}$ is isomorphic to a subgroup of $GO_{10}^{-}(2)$, whose order is $2^{21} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 17$. As in case (a), we have $17 \nmid |F(\bar{G})|$. If 11 divides $|F(\bar{G})|$ then, as the centralizer in $GO_{10}^{-}(2)$ of a Sylow 11-subgroup is of order 33 (see ATLAS [1]) we have that $|F(\bar{G})| = 11$. Now $|\bar{G}| \leqslant 110$. Assume that $11 \nmid |F(\bar{G})|$. As in case (a), we get that 35 divides $|F(\bar{G})|$. As an element of order 35 of $GO_{10}^{-}(2)$ is self-centralizing (see ATLAS [1]), we get that $|F(\bar{G})| = 35$ and $|\bar{G}| \leqslant 6 \cdot 4 \cdot 35$. Now, in both cases,

$$|V|^3/|G| \geqslant |U|^{12}/|G| \geqslant 5^{11}/24 \cdot 35 \cdot 2^{10} > 1.$$

Then $(*)$ holds and V contains a regular G-orbit.

We consider now the case in which e is not a power of 2. As we commented earlier, it is sufficient to show that

$$e^{13/2} \leqslant (24)^{1/3} \, |U|^{3e/8 - 1}. \qquad (**)$$

But, as $|Z(G)|$ divides $|U| - 1$, we have $|U| \geqslant 8$ in this case. Furthermore, $|U| \geqslant 11$ except, possibly, when e is a power of 7. Then $(**)$ holds when $e \geqslant 28$.

If e is even, then 4 divides e by (ii). Then either e is a prime number p, $e = 20$ or $e = 25$. Suppose first that $e = p$ is a prime number. If $p \geqslant 17$, then $|U| \geqslant 32$ and $(**)$ holds. Thus we will consider the cases $e = 5, 7, 11$, and 13.

Case $e = 5$. Here $\bar{G}$ is isomorphic to a subgroup of $SL(2, 5)$, whose order is $2^3 \cdot 3 \cdot 5$. It is easy to see that $\bar{G}$ is isomorphic to a subgroup of Q_8 since $5 \nmid |F(\bar{G})|$ by Remark (1). Next we show that V contains a regular G-orbit. Observe that G has exactly 30 noncentral subgroups of order 5 and at most 50 noncentral involutions. We have

$$\left| \bigcup_{1 \neq x \in G} C_V(x) \right| \leqslant (30 + 50) \, |V|^{5/8} < |V| \qquad \text{since} \quad |V| \geqslant 11^5.$$

Case $e = 7$. Here $\bar{G}$ is isomorphic to a subgroup of $SL(2, 7)$, whose order is $2^4 \cdot 3 \cdot 7$. Then $F(\bar{G})$ is isomorphic to a subgroup of Q_{16}. As G is a $3'$-group, then $\bar{G}$ itself is a subgroup of Q_{16}. Now G contains exactly 56 noncentral subgroups of order 7 and at most 98 noncentral involutions. Now

$$\left|\bigcup_{1\neq x\in G} C_V(x)\right| \leqslant (56+98)\,|V|^{5/8} < |V| \qquad \text{since} \quad |V| \geqslant 29^7.$$

Case $e=11$. Here $\bar{G}$ is isomorphic to a subgroup of $SL(2, 11)$, whose order is $2^3 \cdot 3 \cdot 5 \cdot 11$. If 5 divides $|F(\bar{G})|$, then, as a Sylow 5-subgroup of $PSL(2, 11)$ is self-centralizing, we get that $|F(\bar{G})|$ divides 10. As a Sylow 2-subgroup of $\bar{G}$ is isomorphic to a subgroup of Q_8, we have $|\bar{G}| \leqslant 20$. If $5 \nmid |F(\bar{G})|$, then $F(\bar{G})$ is a subgroup of Q_8 and $|\bar{G}| \leqslant 8$. Now

$$\begin{aligned}|V|^{3/8}/|G| &\geqslant |U|^{33/8}/20 \cdot 11^2 \cdot |Z(G)| \geqslant |U|^{25/8}/20 \cdot 11^2\\ &\geqslant 23^{25/8}/20 \cdot 11^2 > 1.\end{aligned}$$

Then $(*)$ holds and V contains a regular G-orbit.

Case $e=13$. Here $\bar{G}$ is isomorphic to a subgroup of $SL(2, 13)$, whose order is $2^3 \cdot 3 \cdot 7 \cdot 13$. If $|F(\bar{G})|$ is divisible by 7 then, as a Sylow 7-subgroup of $PSL(2, 13)$ is self-centralizing, we have $|F(\bar{G})| \leqslant 14$ and $|\bar{G}| \leqslant 28$. If $7 \nmid |F(\bar{G})|$, then $F(\bar{G})$ is isomorphic to a subgroup of Q_8 and then $\bar{G} = F(\bar{G})$. Thus $|\bar{G}| \leqslant 8$ in this case. Now

$$\begin{aligned}|V|^{3/8}/|G| &\geqslant |U|^{39/8}/28 \cdot 13^2 \cdot |Z(G)| \geqslant |U|^{31/8}/28 \cdot 13^2\\ &\geqslant 13^{31/8}/28 \cdot 13^2 > 1.\end{aligned}$$

Now we consider the cases $e=20$ and $e=25$.

Case $e=20$. We have that $\bar{G}$ is isomorphic to a subgroup of $\bar{G}_1 \times \bar{G}_2$, where $\bar{G}_1$ is a subgroup of $GO_4^-(2)$ and $\bar{G}_2$ is a subgroup of $SL(2, 5)$. The cases $e=4$ and $e=5$ yield that $|\bar{G}_1| \leqslant 20$ and $|\bar{G}_2| \leqslant 8$. Thus $|\bar{G}| \leqslant 160$. Now

$$\begin{aligned}|V|^{3/8}/|G| &\geqslant |U|^{3 \cdot 20/8}/160 \cdot 20^2 \cdot |Z(G)| \geqslant |U|^{52/8}/160 \cdot 20^2\\ &\geqslant 11^{52/8}/160 \cdot 20^2 > 1.\end{aligned}$$

Hence $(*)$ holds and V contains a regular G-orbit.

Case $e=25$. In This case , $\bar{G}$ is isomorphic to a subgroup of the symplectic group $Sp_4(5)$. If 13 divides $|F(\bar{G})|$, then, as a Sylow 13-subgroup of $S_4(5)$ is self-centralizing, we get $|F(\bar{G})| \leqslant 2 \cdot 13$ and $|\bar{G}| \leqslant 26 \cdot 4$. If 13 does not divide $|F(\bar{G})|$, then $F(\bar{G})$ is a 2-group. As $13 \nmid 2^i - 1$ for $1 \leqslant i < 6$, we get that $13 \nmid |\bar{G}|$. Thus $|\bar{G}| \leqslant 2^7 \cdot 5^4$. Anyway $|\bar{G}| \leqslant 2^7 \cdot 5^4$ and

$$\begin{aligned}|V|^{3/8}/|G| &\geqslant |U|^{3 \cdot 25/8}/2^7 \cdot 5^4 \cdot 5^2 \cdot |Z(G)| \geqslant |U|^{67/8}/2^7 \cdot 5^6\\ &\geqslant 11^{67/8}/2^7 \cdot 5^6 > 1.\end{aligned}$$

Then $(*)$ holds and V contains a regular G-orbit. This finishes the proof of Theorem 2.1.

3. Proof of the Main Theorem

HYPOTHESIS 3.1. (i) *G is a solvable $3'$-group.*

(ii) *$F(G) = T * E$, where each Sylow subgroup of E is extraspecial (it is possible that $E = 1$) and the Sylow subgroups of T are cyclic, quaternion, dihedral, or semidihedral. T has as cyclic subgroup U of of index at most 2. We have $U \geqslant Z(F(G))$. The subgroups T, E, and U are normal in G.*

(iii) *Let Z be the subgroup generated by the elements of prime order of U. Put $A = C_G(Z)$. Then $E/Z(E)$ is a faithful and completely reducible $A/F(G)$-module.*

THEOREM 3.2. *Let G be a group satisfying Hypothesis 3.1. Let V be a KG-module, $3 \neq \mathrm{char}(K) \nmid |G|$, such that $[V, u] = V$ if $1 \neq u \in U$. Let F be the algebraic closure of K. Then there exists $v \in V$ such that $W_{C_G(v)}$ is a permutation module for each irreducible constituent W of $V \otimes FG$.*

Proof. Induction on $|G| + \dim_K(V)$. Put $G_1 = C_G(U)$. As G_1 is normal in G, we have $F(G_1) = F(G) \cap G_1$. Now $F(G_1) = U * E$. As $[V, u] = V$ if $1 \neq u \in U$, we have that each irreducible KG_1-submodule of V is faithful. As G_1 is normal in G, then the G_1-module $E/Z(E)$ is completely reducible. Put $V = V_1 \oplus \cdots \oplus V_t$, where each V_i is an irreducible G_1-module. The hypotheses hold for G_1 acting on V_i. Assume $G \neq G_1$. Let v_i be a vector of V_i such that $(W_i)_{C_{G_1}(v_i)}$ is a permutation module for each irreducible constituent W_i of $V_i \otimes FG_1$. Put $v = (v_1, ..., v_t)$. Let W be an irreducible constituent of $V \otimes FG$. As $[V, u] = V$ if $1 \neq u \in U$ and F is algebraically closed, we get that the normalizer of any homogeneous component of W_U is G_1. Then $W = \bar{W}^G$, where $\bar{W}$ is an irreducible G_1-module. But, for each x in G, the module $\bar{W}^x$ is a constituent of $V_i \otimes FG_1$ for some i. Hence $(\bar{W}^x)_{C_{G_1}(v_i)}$ is a permutation module. Let R be a system of coset representatives of the double cosets $\{G_1 x C_G(v) \mid x \in G\}$. By Mackey we have

$$W_{C_G(v)} = \bigoplus_{x \in R} (\bar{W}^x_{G_1 \cap C_G(v)})^{C_G(v)}.$$

As $C_{G_1}(v) \leqslant C_{G_1}(v_i)$, then $\bar{W}^x_{G_1 \cap C_G(v)}$ is a permutation module. Now $W_{C_G(v)}$ is a permutation module. Hence we may assume that $U \leqslant Z(G)$. Thus $U = Z(G) = Z(F(G))$.

Suppose now that E contains an extraspecial subgroup X of order 8 normal in G. As G is a $3'$-group and $E/Z(E)$ is a completely reducible G-module, then we have $G = XC_G(X)$. Now G contains a normal subgroup Y cyclic of order 4 which is not contained in $Z(G)$. Put $C = C_G(Y)$. Suppose that $V = V_1 \oplus \cdots \oplus V_k$, where each V_i is C-irreducible. As C is normal in G, then $F(C) = F(G) \cap C$. It is easy to see that $F(C)$ is the central product of $Z(G)\,Y$ and a group having extraspecial Sylow subgroups (in fact, $C_E(X)$). Put $G_i = C/C_{Z(G)\,Y}(V_i)$. Then, as $Z(G)\,Y$ is central in C, we have that $F(G_i)$ is the image of $F(C)$. Now $F(G_i) = Z(G_i) * E_i$, where each Sylow subgroup of E_i is extraspecial. As C is normal in G, then $E_i/Z(E_i)$ is a completely reducible C-module. As $X/Z(X)$ is a trivial $G/F(G)$-module, then $E_i/Z(E_i)$ is a faithful $G_i/F(G_i)$-module. Clearly $[V_i, z] = V_i$ if $1 \neq z \in Z(G_i)$. Thus the hypotheses hold for G_i acting on V_i. As $G \neq G_i$, then the conclusion holds. Take $v_i \in V_i$ such that $(W_i)_{C_{G_i}(v_i)}$ is a permutation module for each constituent W_i of $V_i \otimes FG_i$. Obviously $(W_i)_{C_c(v_i)}$ is a permutation module, too. Put $v = (v_1, ..., v_n)$. As $[V, y] = V$ if $1 \neq y \in Y$, the same argument as in the first paragraph of the proof yields our claim. Hence E does not contain any extraspecial subgroup of order 8 normal in G. Now Theorem 2.1 finishes the proof of the theorem.

MAIN THEOREM. *Let G be a solvable $3'$-group and let V be a faithful KG-module, where* $\mathrm{char}(K) \neq 3$ *and* $\mathrm{char}(K) \nmid |G|$. *Then there exists $v \in V$ such that $V_{C_G(v)}$ is a permutation module.*

Proof. Induction on $|G| + \dim_K(V)$. First we show that V is irreducible. If not, put $V = V_1 \oplus V_2$. The action of $G/C_G(V_i)$ on V_i yields a vector v_i such that $(V_i)_{C_G(v_i)}$ is a permutation module. Put $v = (v_1, v_2)$. Now $(V_1)_{C_G(v)}$ and $(V_2)_{C_G(v)}$ are permutation modules. Then $V_{C_G(v)}$ is a permutation module, as claimed.

Now we show that V is a primitive KG-module. If not, $V = W^G$, where W is an irreducible KH-submodule of V, $H \neq G$. The inductive hypothesis applies to $H/C_H(W)$ acting on W. We may find a vector w in W such that $W_{C_H(w)}$ is a permutation module. Let $\{x_1, ..., x_t\}$ be a system of coset representatives of $\{Hx \mid x \in G\}$. Put $v = w \otimes x_1 + \cdots + w \otimes x_t$. We may suppose that $\{x_1, ..., x_k\}$ is a system of representatives for the double cosets $\{HxC_G(v) \mid x \in G\}$. By Mackey, we have

$$V_{C_G(v)} = \bigoplus_{i=1}^{k} (W^{x_i}_{H^{x_i} \cap C_G(v)})^{C_G(v)}.$$

But $H^{x_i} \cap C_G(v) \leqslant C_{H^{x_i}}(w \otimes x_i)$. Then each summand in the previous formula is a permutation module and $V_{C_G(v)}$ is a permutation module, as claimed.

Hence V is a primitive KG-module. Now every normal abelian subgroup of G is cyclic. Now Lemma 2.3 of [11] ensures that G satisfies Hypothesis 3.1. As every subgroup of U is normal in G and V is quasiprimitive, we have $[V, u]=V$ if $1 \neq u \in U$. Now Theorem 3.2 applies and yields $v \in V$ such that $(V \otimes FG)_{C_G(v)}$ is a permutation module. The Noether–Deuring theorem (see Theorem VII.1.22 of [5]) now ensures that $V_{C_G(v)}$ is a permutation module.

4. Some Examples

First we show that our theorem is not true for arbitrary solvable groups.

Example 4.1. Consider $G=(C_3 \ltimes Q_8) \times C_3$, where the action of C_3 on Q_8 is faithful. Now G acts faithfully on $V=C_7 \times C_7$. There are 12 noncentral subgroups of order 3 and exactly eight having nontrivial centralizers in V. Then $C_G(v)$ is cyclic of order 3 for each $v \in V-\{0\}$. Clearly $V_{C_G(v)}$ is not a permutation module.

The proof of our theorem is greatly shortened using Lemma 2.4. The next example shows that we must enlarge 5/8 to 3/4 in general (see the proof of Proposition 4 of [4]).

Example 4.2. Consider the symmetric group S_3 acting on $P=D_8 * D_8$ and normalizing two elementary abelian subgroups E_1 and E_2 of order 4 with $E_1 \cap E_2=1$. Consider $G=S_3 \ltimes P$. Clearly an involution x in S_3 is good. Let ϑ be a faithful irreducible complex character of G of degree 4. By Lemma 2.2, we have $|\vartheta(x)|=2$. As x is an involution, we have $\vartheta(x)=\pm 2$. Now

$$\dim_{\mathbb{C}}(C_V(x))=\tfrac{1}{2}(\vartheta(x)+4).$$

If $\vartheta(x)=2$, then $\dim_{\mathbb{C}}(C_V(x))=3=3 \dim_{\mathbb{C}}(V)/4$. If $\vartheta(x)=-2$, consider $\vartheta\lambda$, where λ is the nontrivial linear character of G. Now $(\vartheta\lambda)(x)=2$ and our claim is verified.

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